A STUDY OF SUPER OPERATORS

By

Saiyid Abutalib Nizami

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By

Professor M.A.Kazim

HEAD

Department of Mathematics and Statistics

A. M. U. Aligarh.

Certified
M. A. Kazim

Head
Deptt. of Maths & Stat.
A. M. U. ALIGARH.

S. Abu Talib Nizami.

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" A STUDY OF SUPER OPERATORS "

PREFACE

This thesis entitled " A study of Super Operators" is the research work done by me since 1960 in the department of Mathematics and Statistics, A.M.U. Aligarh, Regional Engineering College, Srinagar, Kashmir and Annamalai University. This work was at intervals financially supported by a Fellowship at Aligarh and a travel and research grant by Kashmir University.

The theory of completely continuous linear operator has been developed by Hilderhandt, Maddaus, Phillips, Grothendick, Taam and several others for many years. Our aim in the thesis is two-fold. Firstly we study the well known open problem of the representation of a completely continuous linear operator as a uniform limit of finite dimensional operators. We have investigated the problem (i) from F-space to Banach space of "type A" (ii) from Banach space to F-space of " type A" and (iii) from F-space to F-space of " type A" . We have also obtained equivalent conditions of hyponormality and investigated the conditions under which a complex Hilbert space has an orthonormal Schauder basis comprising of eigen vectors of the operator on the space. Secondly, we have introduced a new concept of Super operator on F-spaces (which we also call S-operator) This is an extension of the concept of strictly singular operator the theory of which has been developed as an extension of completely continuous operator within last twelve years by Kato, Goldberg, Thorp, Whitely, Lacey, Bessage, Klee and Pelesynski.

This super operator we have tried to study in several directions like its existence, its seperability, its conjugate its spectrum property and its relation with reflexivity and almost reflexivity etc.

The techniques used are sometimes analogous and at places new and original.

AT the outset I must acknowledge my deep indebtedness to Professors Kato, Goldberg, Thorp, Whitely, Lacey, Wilansky, Bessage and Klee whose works encouraged me to introduce the concept of S-operator and develop its theory. My obligations also go to Professors Hilderhandt, Maddaus, Phillips, Taam, Wilansky and all others whose works encouraged me to study the problem of representation of completely continuous operator in some uninvestigated directions. My gratitude also goes to Professor Taqdir Husain of the McMaster University, Canada and Professor V.Ganapathy Iyer, Annamalai University for the suggestions and guidance which I received from them either through correspondence or personal contacts.

The main part of this thesis consists of six chapters bordered by a zero chapter of preliminaries statements and remarks which we use later, and by an appendix in which we reproduce my paper entitled "A Note on Hurwitz's Theorem" published in "Mathematics Student" Vol. XXXIV No.2 April-June 1966.

The first chapter denotes to the solution of an open problem of representation of completely continuous linear operator on F-spaces. We also study equivalent conditions of hyponormality.

In the second chapter we introduce the concept of Super operator and give four existence theorems of an S-operator.

The third chapter deals with the seperability of an S-operator. The fourth chapter establishes a connection between S-operator, reflexivity and almost reflexivity. The fifth chapter gives the ~~property~~ S-operators with their conjugates. The last chapter studies the spectrum property.

Lastly I express my respects and gratitude to my supervisor Prof. M.A.Kazim for his careful going through the thesis, valuable suggestions in its improvement and consequent help during its preparation. I also thankfully acknowledge the keen interest shown by Mr. Mukhtar Nabi Khan Department of Mathematics and Statistics in typing the thesis. I also thank Mr. Rashid Ahmad, the incharge Seminar Library of the department for providing facilities of an easy access to research material. My thanks also go to all persons and agencies who have been of some help and support during the preparation of this work.

Abu Talib Nizami

Saiyid Abu Talib Nizami.

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CONTENTS

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SYMBOLS AND NOTATIONS

Σ	:	the summation
R^n	:	n-tuples of real numbers a
$\langle s \rangle$	:	the space of all sequences
$C[a,b]$	:	the space of continuous function on $[a,b]$
$C(E)$	:	the space of continuous functions on E, E being a Hausdorff space.
L^p	:	the space of Lebesgue measurable functions f on $[0,1]$ such that $\int_0^1 \|f\|^p < \infty$
X^*	:	conjugate space of X or Dual space of X
X^{**}	:	second dual of X
$X^* \otimes X$	:	Tensor product of X^* and X
T^*	:	Adjoint Operator of T
T^{**}	:	Adjoint Operator of T^*
$L_C(X,Y)$	:	the space of continuous linear operators on X to Y.
D	:	the space of infinitely differentiable functions on R^n.
$N(\lambda - T)$	:	Null space of $\lambda - T$

F - space	:	Frechet space
R(T)	:	the range of T
X	:	the completion of X
$\sigma(T)$	:	the spectrum of T
$\pi(T)$	:	the approximate spectrum of T.
$P_\sigma(T)$	:	the point spectrum of T
$\rho(T)$	:	the resolvent set of T
$C_\sigma(T)$	:	the continuous spectrum of T
$R_\sigma(T)$	:	the residual spectrum of T
X^+	:	the set of linear functionals on X
l_p	:	the space of p^{th} power summable sequences i.e. $x \in l_p$ $x = x_1, x_2, \ldots, x_n, \ldots$ such that $\sum_{n=1}^{\infty} \lvert x_n \rvert^p < \infty$

Note:- Bibliography is arranged alphabatically authorwise and numbered according to a paper has appeared. Definitions, Theorems, and corollaries etc have been arranged chapterwise.

CHAPTER - 0

PRELIMINARIES

In this chapter we mention all important basic concepts, certain remarks and known results which we want to use in this thesis.

0.1 Basic concepts :

Definition 0.1 Linear space : A linear space X is a set for which are defined (i) the operation of addition making X an abelian group, (ii) an operation of scalar multiplication over a field of scalars satisfying the following laws :

(i) $\lambda(x+y) = \lambda x + \lambda y$,

(ii) $(\lambda + \mu)x = \lambda x + \mu x$, where λ, μ are scalars

(iii) $(\lambda\mu)x = \lambda(\mu(x))$, and $x, y \in X$

(iv) $1x = x$.

Definition 0.2 Paranorm : A paranorm is a real function p defined on a linear space X, satisfying the following conditions for all vectors x, y.

(i) $p(0) = 0$,

(ii) $p(x) \geq 0$,

(iii) $p(-x) = p(x)$,

(iv) $p(x+y) \leq p(x) + p(y)$, [the triangular inequality]

(v) If λ_n is a sequence of scalars with $\lambda_n \to \lambda$ and x_n is a sequence of vectors with limit x, then $p(\lambda_n x_n - \lambda x) \to 0$ [continuity of multiplication]

Definition 0.3 <u>Weak and strong paranorm</u> : For any two paranorms p,q defined on linear space X, p is stronger than q and q is weaker than p, if, whenever x_n is a sequence in X such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each is stronger than the other, p and q are said to be equivalent otherwise one is strictly stronger and one is strictly weaker.

Remark 0.1 A paranorm induces the semi-metric on the space.

Definition 0.4 <u>Total paranorm</u> : A paranorm for which $p(x) = 0 \iff x = o$, is called <u>total</u> paranorm.

Remark 0.2 A total paranorm induces the metric on the space.

Definition 0.5 <u>Semi-norm</u> : A semi-norm is a real function p defined on a linear space X, satisfying for all vectors x,y, scalars λ

(i) $p(x) \geq 0$,

(ii) $p(\lambda x) = |\lambda|\, p(x)$ [homogeneity]

(iii) $p(x+y) \leq p(x) + p(y)$, [the triangular inequality]

Definition 0.6 <u>Norm</u> : A norm is a real function p, defined on a linear space X, satisfying for all vectors x,y and scalar

(i) $p(x) = 0$ iff $x = \theta$, $p(x) > 0$ if $x \neq o$

(ii) $p(\lambda x) = |\lambda|\, p(x)$, [homogeneity]

(iii) $p(x+y) \leq p(x) + p(y)$ [the triangular inequality]

Note : The norm of x is also written as $\|x\|$. In the above definition, therefore $p \sim \| \ \|$.

Remark 0.3 The norm induces the usual metric

$$d(x,y) = \| x - y \| = p(x - y)$$

Definition 0.7 Total set : A set Φ of functions from a linear space X to a linear space is total (or total over X) if given $x \in X$, there exists $f \in \Phi$ such that $f(x) \neq 0$ or $f(x) = 0$ for all $f \in \Phi \Longrightarrow x = 0$.

Definition 0.8 Inner product : An inner product on a linear space X is a complex valued function denoted by (x,y), $x,y \in X$, satisfying the following conditions for any $x,y,z \in X$, λ, μ being scalars

(i) $\overline{(x,y)} = (y,x)$, [the complex conjugate]

(ii) $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$, [Linearity]

(iii) $(x,x) \geq 0$ for all $x \in X$ and equal to zero if and only if x is the zero vector.

Remark 0.4 The inner product induces the norm denoted by $\|x\|^2 = (x,x)$ on the space.

Definition 0.9 Semi-metric : A semi-metric on X is a function d from $X \times X \to R$, satisfying for all $x,y,z, \in X$

(i) $d(x,y) = 0$ iff $x = y$,

(ii) $d(x,y) = d(y,x) > 0$, [symmetry]

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ [the triangular inequality]

Remark 0.5 The set X together with the semi-metric d is called a semi-metric space and denoted by (X,d).

Definition 0.10 Metric : A metric on X is a function d from $X \times X \to R$ satisfying for all $x,y,z \in X$

(i) $d(x,y) = 0$ iff $x = y$,

(ii) $d(x,y) = d(y,x) > 0$, [symmetry]

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ [the triangular inequality]

Remark 0.6 (X,d) is called the metric space associated with metric d.

Definition 0.11 Homeomorphism: A function $f : X \to Y$ is called a homeomorphism from a linear space X into a linear space Y if it is one-to-one and continuous and $f^{-1} : f(X) \to X$ is also continuous, homeomorphism will be onto if, in addition, f is onto. In this case we say that X,Y are homeomorphic.

Definition 0.12 Topology : A topology for a set X is a family $\mathcal{T} = \{O_i\}$ of subsets of X, satisfying the following axioms

(i) $\phi \in \mathcal{T}$,

(ii) $X \in \mathcal{T}$,

(iii) $\cup O_i \in \mathcal{T}$,

(iv) $\bigcap_{k=1}^{n} {}_{k} \in \mathcal{T}$.

Note : The set X with the topology $\mathcal{T}$ on X is a topological space and is denoted by $(X, \mathcal{T})$. The members O_i of are called <u>open sets</u> and the elements of X are called points of the topological space.

Remark 0.7 If several tpologies for X are being considered, then the members of $\mathcal{T}$ will be called $\mathcal{T}$- open or open relative to $\mathcal{T}$ and a similar convention will hold also for other topological objects to be defined.

Definition 0.13 <u>Linear topological space</u>: X is called a linear topological space if X is both a linear space and a topological space. Similarly X is called a linear metric (semi-metric) space if X is both a linear space and a metric (semi- metric) space.

Definition 0.14 <u>Linear map</u> (linear operator) T is said to be a linear map also called linear operator on a linear topological space X to a linear topological space Y, if for every $x_1, x_2 \in X$ and scalar λ, satisfying

(i) $T(x_1 + x_2) = Tx_1 + Tx_2$,

(ii) $T(\lambda x_1) = \lambda Tx_1$.

Note (1) A linear map is also called a homomorphism.

(2) If T is a linear map on a space X to itself, it is an endomorphism.

Definition 0.15 : A one-to-one linear map on a linear topological space X onto a linear topological space Y is called linear isomorphism. A topological isomorphism is a linear isomorphism which is also a homeomorphism.

Definition 0.16 Neighbourhood : Let $(X, \mathcal{T})$ be a topological space. A set V is said to be a neighbourhood ($\mathcal{T}$-neighbourhood) of a point $x \in X$ if there exists a set $U \in \mathcal{T}$ such that $x \in U \subseteq V$.

Definition 0.17 Covering (Open covering) : Let I be an indexing set and $\{G_\alpha\}_{\alpha \in I}$ be a family of sets (open sets), further suppose that the family has the property that $X \subseteq \bigcup_{\alpha \in I} G_\alpha$. We say that G_α is a coverning (an open coverning) of X.

Definition 0.18 Compactness : If the space X has the property that for every open covering one can select a finite open subcovering, symbolically

$$X \subseteq \bigcup_{\alpha \in I} G_\alpha \implies X \subseteq \bigcup_{k=1}^{n} G_{\alpha_k}$$

then X is called compact.

Definition 0.19 Basis for a topological space : Let $(X, \mathcal{T})$ be a topological space. A collection B of open sets such that every $\mathcal{T}$-open set can be written as a union of sets from B is called a basis for $\mathcal{T}$.

Definition 0.20 : <u>First axiom of countability</u> : A topological space $(X, \mathcal{T})$ is said to satisfy the first axiom of countability if at each point $x \in X$, there exists a countable basis for $\mathcal{T}$.

Remark 0.8 : If a topological space X satisfies the first axiom of countability, then " sequences" may be used to characterize compactness, countable compactness, sequential compactness, completion, and other notions.

Definition 0.21 <u>Countable compactness</u> : A space X is countably compact if each countable open covering has a finite open subcovering.

Note : A space X is countably compact if and only if each sequence in X has a limit in X.

Definition 0.22 <u>Sequential compactness</u> : A space X is sequentially compact if each sequence has a subsequence which converges to a point in X.

Definition 0.23 <u>Relative compactness</u> : A space X is said to be relatively compact if $\bar{X}$ (the closure of X) is compact; or a set U of X is said to be relatively compact if $\bar{U}$ is compact.

Definition 0.24 : <u>Completion</u> : A space X is said to complete if every Cauchy sequence in X is convergent in X.

Note : X is a space, its completion is denoted by $\widehat{X}$.

Definition 0.25 Precompactness : A space X is said to be precompact if $\hat{X}$, the completion of X is compact.

Definition 0.26 : Totally bounded set : Let U be a set of a metric space (X,d). If from every sequence of points from U one can select a convergent subsequence, then U is called totally bounded.

Remark 0.9 : In a metric space (X,d), (i) a set U is relatively compact, it is also totally bounded, (ii) further if (X,d) is complete and U is totally bounded, then U is relatively compact.

Definition 0.27 Circled set : A set U of linear space X is called circled if $\lambda U \subseteq U$ for scalars λ satisfying $|\lambda| \leq 1$.

Definition 0.28 : Absorbing set : A set U of a linear space X is called absorbing if for each vector $x \in U$, there exists $\epsilon > 0$, such that $\lambda x \in U$ whenever λ is a scalar satisfying $|\lambda| \leq \epsilon$.

Definition 0.29 Dense set : A subset U of X is said to be dense in X if the closure of U, $\overline{U} = X$.

Definition 0.30 Nowhere dense set : A subset U of X is said to be nowhere dense in X if $c\overline{U}$ (the complement of $\overline{U}$) is dense in X or that $\overline{c\,\overline{U}} = X$

Definition 0.31 <u>Sets of I-category and II-category</u> : A set U is said to be of I-category if it can be written as a countable union of nowhere sets. Otherwise it is said to be of II-category.

Definition 0.32 <u>Separability</u> : A space X is said to be separable if it contains a countable dense subset.

Definition 0.33 <u>Schauder basis</u> : A schauder basis for a linear metric space is a sequence $\{x_n\}$ such that for any vector x there exists a unique sequence $\{\lambda_n\}$ of scalars such that $\sum_{n=1}^{\infty} \lambda_n x_n = x$. The series $\Sigma \lambda_n x_n$ which converges to x is called the expansion of x .

Definition 0.34 <u>Analytic function</u> : A function f(z) of complex variable z is said to be analytic in a domain D, if it is one valued and differentiable at every point and in the neighbourhood of every point of D except a finite (infinite) number of points. These points are called the singularities of the function f(z) in D .

Note : If there is no singularity of f(z) in D, then f(z) is called <u>regular</u> in D.

Definition 0.35 <u>Convex set</u> : A set U in X is called convex if $\lambda U + \mu U \subseteq U$ for all pairs of scalars λ, μ, satisfying, $\lambda \geq 0$, $\mu \geq 0$, and $\lambda + \mu = 1$.

Definition 0.36 Strong limit (limit in norm): Let $\{T_k\}$ be a sequence of continuous operators in $L_C(X,Y)$, T_k is said to converge to T (continuous operator) uniformly, if given $\epsilon > 0$ there exists a positive integer K such that, for all $k > K$,

$$\| T_k - T \| < \epsilon$$

Definition 0.37 Convex body : A convex set U with non-empty interior is called a convex body.

Definition 0.38 Complementary spaces : Two spaces U and V of X are complementary if and only if $X = U \oplus V$, in other words if each member of X can be written in one and only one way as the sum of a member of U and a member of V.

Definition 0.39 Co-dimension (deficiency) : The co-dimension (deficiency) of a sub space U of a space X is defined to be the dimension of a subspace of X which is complementary to U in X.

Definition 0.40 Linearly independent set : A set of vectors $x_1, x_2, \ldots, x_n$ in a linear space is said to be linearly independent if the relation

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$$

implies that each $\lambda_i = 0$. An arbitrary set of vectors is said to be linearly independent if every finite subset is linearly independent.

Definition 0.41 <u>Hamel dimension</u> : A Hamel basis for a linear space X is a linearly independent set which spans X and any two Hamel basis for the space X are in one-to-one correspondence. Now we define the Hamel dimension of X to be the cardinality of its Hamel basis.

Note : The Hamel dimension of a set in a linear space X is defined to be the Hamel dimension of its span.

Definition 0.42 <u>Orthogonal complement</u> : Let X be an inner product space and let U be a subset of X. The collection of vectors

$$U^{\perp} = \{y \in X \text{ such that } y \perp x \text{ for all } x \in U\}$$

is called the orthogonal complement of U.

Definition 0.43 <u>Orthogonality</u> : Let X be an inner product space and let $x, y \in X$. Then x is said to be orthogonal to y, written $x \perp y$, if $(x,y) = 0$.

Definition 0.44 <u>Point-wise discontinuous function</u> : A function $f(x,y,t)$ defined in three dimensional space R^3 is called point-wise discontinuous if the set of points of continuities forms a set dense in the three dimensional space R^3.

Definition 0.45 <u>Directed sequence</u> : A binary relation $\geq$ directs a sequence $\{x_n\}$ if $\{x_n\}$ is nonempty and

(i) If $x_1 \geq x_2$ and $x_2 \geq x_3$, then $x_1 \geq x_3$ [transitivity]

(ii) $x_1 \geq x_1$ [reflexivity]

(iii) If x_m, x_p are the elements of the sequence, then there is x_q in the sequence such that

$x_q \geq x_m$ and $x_q \geq x_p$ [special character]

A directed sequence $\{x_n\}$ is a pair $(\{x_n\}, \geq)$ such that $\geq$ directs $\{x_n\}$.

0.2 Abstract Spaces :

Definition 0.46 Locally convex space : We call a linear topological space X, a locally convex space, if, each neighbourhood of 0 includes a convex neighbourhood of 0.

Definition 0.47 F-space or Frechet Space : A complete locally convex linear metric space X is called F-space or Frechet space.

Definition 0.48 Hausdorff space : A topological space in which different points have disjoint neighbourhoods is called a Hausdorff space.

Definition 0.49 Barrel : Let X be a locally convex space. A subset U of X which is circled convex, closed and absorbing is called a barrel.

Definition 0.50 Barrelled space : A locally convex space X is called barrelled space or t-space if every barrel in X is a neighbourhood of 0.

Definition 0.51 Montel space : A Hausdorff space in which every closed bounded set is relatively compact is a Montel space.

Definition 0.52 : A semi-metric space which is also linear space is called a linear semi-metric space, a linear metric space, a semi-normed space or a normed space, if the semi-metric comes from a paranorm, a total paranorm, a semi-norm or a norm respectively.

Definition 0.53 Inner product space : A linear space X with an inner product defined on it will be called an inner product space.

Definition 0.54 Hilbert space : If an inner product space X is complete with respect to the metric derived from the inner product, X is said to be a Hilbert space.

Definition 0.55 Normed space : A linear space X with a norm defined on it will be called a normed linear space.

Definition 0.56 Banach space : If a normed linear space X is complete with respect to the metric $d(x,y) = \| x - y \|$ for every $x,y \in X$, derived from the norm, X is said to be a Banach space.

Definition 0.57 ''type A space'' . By a space of ''type A'' we shall mean a Banach space Y in which there exists a linearly independent sequence $\{y_s\}$ of elements of unit norm and a double sequence $\{L_{rs}(y)\}$ of continuous linear operators such that for every $y \in Y$

$$\lim_{r \to \infty} \| y - \sum_{s=1}^{r_s} L_{rs}(y)\, y_s \| = 0$$

Definition 0.58 Subprojective space : A normed space Y is subprojective, if, given any closed infinite dimensional subspace M of Y, there exists a closed infinite dimensional subspace N contained in M and a continuous projection ofY onto N.

Note(i): The definition can be extended to non-normable F-spaces.

Note(ii): Throughout the thesis, we shall take X, Y, X^*, Y^*, X^{**} and Y^{**} as infinite dimensional non-normable F-spaces.

0.3 Operators :

Definition 0.59 Continuous linear operator : Let T be a linear operator on a normed linear space X to a normed linear space Y, $T : X \to Y$, and let $\{x_n\}$ be a sequence in X, $\{Tx_n\}$ will be a sequence in Y, T is said to be continuous if $x_n \to x$, $x \in X$ implies $Tx_n \to Tx$, $Tx \in Y$.

Definition 0.60 Bounded linear operator : Let T be a linear operator on a normed linear space X to a normed linear space Y, $T : X \rightarrow Y$,

T is said to be bounded if there exists some positive constant K such that, for all $x \in X$,

$$\| Tx \| \leq K \| x \|$$

Remark 0.10 : A Linear operator T on a normed linear space X to a normed linear space Y is continuous if and only if it is bounded.

Definition 0.61 Completely continuous linear operator : (Compact Operator) A continuous linear operator T on a normed linear space X to a normed linear space Y, is called completely continuous (compact operator) if it maps bounded sets (of X) into relatively compacts (of Y).

Definition 0.62 Finite dimensional operator (linear operator with finite range): A linear operator $T : X \rightarrow Y$ is called finite-dimensional (linear operator with finite range) if T(X) is finite dimensional.

Definition 0.63 Inverse operator : Let T be a continuous linear operator on a normed linear space X. The inverse operator of T, T^{-1} is said to exist if there exists a function T^{-1} such that for every $x \in X$, $T^{-1}(Tx) = x$. The domain of T^{-1} is just the range of T, R(T), and its range is X:

$$T^{-1} : R(T) \rightarrow X .$$

Note : Please note that the inverse of T^{-1} is just T.

Definition 0.64 : Strictly singular operator : A continuous linear operator T from a normed linear space X to another normed linear space Y is said to be strictly singular if, whenever the restriction of T to a subspace U of X has a continuous inverse, U is finite dimensional. If words " a subspace U" are replaced in the definition by the words " a closed subspace U", the new definition is equivalent to the original one, because a finite dimensional space is closed.

Definition 0.65 Projection operator : Suppose one has a direct sum decomposition of the space X, $X = U \oplus V$. Any $z \in X$ can now be uniquely written $z = x + y$, when $x \in U$ and $y \in V$. The operator T defined by $Tz = x$ is called the projection operator on U along V.

Definition 0.66 Orthogonal projection operator : An orthogonal projection operator is defined to be projection operator on U along the orthogonal complement $U^{\perp}$ of U.

Definition 0.67 Adjoint operator : Let T be a linear operator on a finite dimensional inner product space X, $T : X \to X$, then T^* is a linear operator on X, $T^* : X \to X$, called the adjoint operator of T defined by the equation

$$(Tx,y) = (x,T^*y), \qquad (\text{ for all } x,y \in X)$$

Note : This definition of adjoint operator can be extended to Hilbert spaces.

Definition 0.68 Self-adjoint and normal operators : Let T be a linear operator on a finite dimensional inner product space X if

(i) $T = T^*$, then T is said to be self-adjoint operator

(ii) $T T^* = T^* T$, then T is said to be normal operator.

Note : The definition of self-adjoint and normal operators can be extended to Hilbert spaces.

Definition 0.69 Hyponormal operator : Let X be a complex infinite dimensional inner product (Hilbert space) and let T be a continuous linear operator on X, $T : X \to X$. Then T is called hyponormal operator if $\| T^*x \| \leq \| Tx \|$.

Definition 0.70 Idempotent operator : An operator on a linear space X, $T : X \to X$, is called idempotent if $T^2 = T$, where T^2 we mean the operator taking x into $T(Tx)$, $T^2x = T(Tx)$.

0.4 Notions related to spaces :

Definition 0.71 Pointwise boundedness : Let X,Y be semi-normed spaces, and Φ a family of functions $f : X \to Y$. Then Φ is said to be point-wise bounded if for each $x \in X$, the set $\{ f(x) \mid f \in \Phi \}$ is bounded set in Y.

Definition 0.72 <u>Uniform boundedness</u> : Let X,Y be semi-normed spaces and Φ a family of linear functions $f : X \to Y$ or semi-norms defined on X. Then Φ is said to be uniformly bounded, if there exists a positive constant K such that $\|f\| \leq K$ for all $f \in \Phi$.

Definition 0.73 <u>Weak Topology</u> : Let Φ be a total countable family of linear functionals on X. The topology generated by Φ is called the weak topology on X, denoted by $w(X,\Phi)$.

Definition 0.74 <u>Weak linear topology</u> : Let X be a linear space, Φ a family of semi-norms p on X to a collection of linear topological spaces. The weak linear topology generated by Φ, designated $\sigma(X,\Phi)$ or, simply $\sigma(\Phi)$ is the linear topology generated by $\{p^{-1}(G) \mid p \in \Phi$, G a neighbourhood of 0 in the range of p $\}$.

Definition 0.75 <u>G-topology</u> : Let X,Y be two linear topological spaces and G be a class of subsets of X. We define a topology on the set of all continuous linear operators $L_C(X,Y)$ of uniform convergence over sets M in G as follows :

Let M be in G and V be a neighbourhood of O in Y. Let

$$T(M,V) = \{f \in L_C(X,Y) \mid f(M) \subseteq V\}$$

The collection of T(M,V), when M runs over G and V over neighbourhoods of O in Y, forms a sub-basis for a topology called the G-topology. The most important particular cases of G-topology are as follows :

(i) G consists of all finite subsets of X. In this case the topology is called the G- topology of simple convergence.

(ii) G consists of all compact subsets of X. Then the G-topology is called the topology of compact convergence or uniform convergence over compact subsets.

(iii) G consists of all precompact subsets of X. The G-topology is called the topology of precompact convergence.

(iv) G-consists of all bounded subsets of X. The G-topology is called as the uniform convergence topology over bounded sets.

Definition 0.76 <u>Topologies on non-normable F-spaces</u> : Let X,Y be non-normable F- spaces and p_n , q_n be the total sequences of semi-norms which define the topologies on X and Y by p and q respectively, where

$$p(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1 + p_n(x)}, \quad x \in X$$

$$q(y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{q_m(y)}{1 + q_m(y)}, \quad y \in Y$$

Definition 0.77 <u>Semi-norm and metric on A(U)</u> : Let U be an open proper subset of the Riemann sphere (the compactification of the complex plane by the one point ∞). Let A(U) be the space of complex valued functions, analytic on U, and vanishing at ∞ if $\infty \in U$, with the topology of uniform convergence on compact subsets U. We take an increasing sequence $\{U_n\}$ of compact subsets of U whose union U.

Then

$$\operatorname*{lub}_{z \in U_n} |f(z)| = \| f \|_n$$

is a semi-norm (total) on A(U). We define a metric d on A(U) induced by this semi-norm as

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f - g \|_n}{1 + \| f - g \|_n} ,$$

where $f, g \in A(U)$.

Note : A(U) is non-normable F-space and Montel space.

Definition 0.78 <u>Metric on $\mathcal{D}$</u> : Let $\mathcal{D}$ be the space of infinitely differential functions on R^n.

Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be a point of R^n and $\{p\} = \{p_1, p_2, p_3, \ldots, p_n\}$ be a set of n non-negative integers and $|p| = p_1 + p_2 + p_3 + \ldots + p_n$. For each p, define a differential operator

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \, \partial x_2^{p_2} \, \partial x_3^{p_3} , \ldots , \partial x_n^{p_n}}$$

The topology of compact convergence for derivatives on $\mathcal{D}$ is defined by the family of semi norms (total)

$$q_U^{(m)} \mid \text{for each } U \text{ compact, } U \subseteq R^n \quad m = 0,1,2,3,\ldots$$

where

$$q_U^{(m)}(f) = \sup \mid D^p f(x) \mid \ \mid x \in U, \quad 0 \leq |p| \leq m$$

The metric d on $\mathcal{D}$ induced by the above semi-norm is given by

$$d(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \quad \frac{q_U^{(m)}(f,g)}{1 + q_U^{(m)}(f,g)} \ .$$

Note : $\mathcal{D}$ is non-normable F-space and Montel space.

Definition 0.79 <u>Reflexivity on non-normable F-spaces</u> : Let X,Y be non-normable F-spaces and $\{p_n\}$, $\{q_m\}$ be the total sequences of semi-norms which define the topologies X_p and Y_q by p and q as in Definition 0.76

Let U_p be the topology defined by p restriction to U, the subspace of X_p, where $U \subseteq X$. $U^{*\beta}$ is said to be the strong dual of U if β is the G-topology on U, where G consists of p-bounded sets of U. U is said to be reflexive if $U^{*\beta*\beta} = U_p$, where $U^{*\beta*\beta}$ is the strong bidual of U.

Definition 0.80 <u>Weak convergence of a sequence</u> : A sequence x_n in a F-space X is said to converge weakly to x in X if it converges in the weak topology $w(X, \Phi)$, where Φ is total countable family of linear functions on X.

Definition 0.81 Strong convergence of a sequence : A sequence $\{y_n\}$ in a non-normable F-space Y on which the topology induced by a total paranorm q as in [Definition 0.76] is defined by a total sequence $\{q_m\}$ of semi-norms, is said to converge strongly to y in Y if $q(y_n - y) \to 0$, as $n \to \infty$.

Definition 0.82 Weak Cauchy sequence in F-space : A Cauchy sequence in a F-space X is a weak cauchy sequence if it is a cauchy sequence in the weak toplogy $w(X, \Phi)$, where Φ is a total countable family of linear functions on X.

Definition 0.83 : Almost reflexive F-space : A F-space is called almost reflexive if every bounded sequence in X contains a weak cauchy subsequence.

Definition 0.84 Semi-norms and Metrics on X^* and Y^* : If we define semi-norms on X^*, Y^*, and the induced metrics define topologies on X^* and Y^* where X^*, Y^* are strong duals of a non-normable F-space and a non-normable F-space Y respectively. Taking increasing sequence of bounded sets $\{V_n\}$ of X with topology as in [Definition 0.76] whose union is X . Then

$$\underset{x \in V_n,\ f \in X^*}{\text{l.u.b}} |f(x)| \text{ , defines a semi-norm (total)}$$

on X^*.

And taking increasing sequence of bounded sets $\{U_m\}$ of Y with topology as in [Definition 0.76] whose union is Y. Then

$$\operatorname*{l.u.b}_{y \in U_m,\ g \in Y^*} |g(y)| \quad , \quad \text{defines a semi-norm}$$

(total) on Y^*.

We define the metrices d_1 and d_2 on X^* and Y^* respectively by

$$d_1(f_1, f_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f_1 - f_2 \|_n}{1 + \| f_1 - f_2 \|_n} ,$$

where $f_1, f_2 \in X^*$

$$d_2(g_1, g_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\| g_1 - g_2 \|_m}{1 + \| g_1 - g_2 \|_m}$$

which induce the topologies on X^* and Y^* respectively.

Note : X^* and Y^* are non-normable F-spaces.

Remark 0.11 The topology on the second dual, X^{**} of X is induced by the total paranorm p restriction to X^{**}, which is defined by a total sequence $\{p_n\}$ of semi-norms on X^{**} as in [Definition 0.76] and the topology on the second dual Y^{**} of Y is induced by the total paranorm q restriction to Y^{**}, which is defined by a total sequence $\{q_m\}$ of semi-norms on Y^{**} as in [Definition 0.76].

Definition 0.85 <u>Duality</u> : Let X,Y be linear spaces and f a bilinear functional on $X \times Y$ with the property that, for each $x \neq 0$, there exists y such that $f(x,y) \neq 0$ and for each $y \neq 0$, there exists x such that $f(x,y) \neq 0$. Then X,Y are said to be in duality with respect to f.

Definition 0.86 <u>Polar</u> : Let X,Y be linear spaces in duality with respect to f. For a subset U of X, defines U^o, the polar of U by

$$U^o = \{ y \in Y \mid |f(x,y)| \leq 1, \quad \text{for all } x \in U \}$$

A similar definition is given for U^o if U is a subset of Y. Using the abbreviation $\lambda - T$, λ a scalar to $\lambda I - T$, where I represents the identity operator and T and $\lambda - T$ are continuous linear operator on a normed linear space X, we make the following definitions :

Definition 0.87 <u>Resolvent set of T</u> : If the range of $\lambda - T$ is dense in X and if $\lambda - T$ has a continuous inverse on $R(\lambda - T)$, then λ is said to belong to the resolvent set of T, denoted by $\rho(T)$.

Definition 0.88 <u>Continuous spectrum of T</u> : If the range of $\lambda - T$, $R(\lambda - T)$ is dense in X and if $\lambda - T$ has an unbounded inverse (that is, we assume $(\lambda - T)^{-1}$ exists but is not bounded) then λ is said to belong to the continuous spectrum of T, denoted $C_\sigma(T)$.

Definition 0.89 Residual spectrum of T : If the range of $\lambda - T$ is not dense in X but $\lambda - T$ has an inverse, bounded or unbounded, then λ is said to belong to the residual spectrum of T, denoted by $R_\sigma(T)$.

Definition 0.90 Point spectrum of T : If $(\lambda - T)^{-1}$ does not exist and the range of $\lambda - T$ may be dense or non-dense in X, then λ is said to belong to the point (discrete) spectrum of T denoted by $P_\sigma(T)$. This set consists of just eigen values of T.

Definition 0.91 Eigen Value : Let T be a continuous linear operator on a normed linear space X, λ is a scalar, if $(\lambda - T)x = 0$, $x \in X$ and $x \neq 0$, then λ is called the eigen value of T and x is called the eigen vector.
Note : The definition of eigen-value can be extended to a non-normable F-space.

Definition 0.92 Spectrum of T : Let T be a continuous linear operator on a normed linear space X. The set $C_\sigma(T) \cup R_\sigma(T) \cup P_\sigma(T)$ denoted $\sigma(T)$ is called the spectrum of T.

Definition 0.93 Approximate Spectrum of T : Suppose that $T : X \to X$. The scalar λ is called an approximate proper value of T if any $\epsilon > 0$, there exists an $x \in X$ such that $\|x\| = 1$ and $\|(\lambda - T)x\| < \epsilon$. We denote $\pi(T)$ the set of all approximate proper values and call this set the approximate spectrum of T.

0.5 <u>Preliminary results with remarks</u> :

We state below several important results, available in literature, which we shall require during the course of this thesis.

Problem 0.1 : Let T be a completely continuous linear operator on a Banach space X to a Banach space Y. Can T be approximated arbitrarily close in norm by a sequence of continuous linear operators with finite range ?

[Hille and Phillips [57] p 49]

Theorem 0.1 : If T is a completely continuous linear operator and $\{T_n\}$ is a sequence of continuous linear operators with finite range on a Hilbert space X to a Hilbert space Y, then

$$\lim_{n \to \infty} \| T_n - T \| = 0$$

[Riesz and Nagy, [55] p. 204]

Theorem 0.2 : Every completely continuous linear operator on the space C can be made approximately close in norm by a sequence of continuous linear operators with finite range

[Riesz and Nagy, [55] p.223]

Theorem 0.3 : Every completely continuous linear operator on a Banach space X to a space Y of " type A" is the strong limit of a sequence of continuous linear operators with finite range.

[Haddaus, [38] p. 281]

Theorem 0.4 : Every completely continuous linear operator on a Banach space X which is either L^p or C to a Banach space Y is the strong limit of a sequence of continuous linear operators with finite range.

[Phillips, [40] p.530]

Theorem 0.5 : A completely continuous linear operator on a Banach space X to the space C(Ω) can be approximated arbitrarily in norm by a sequence of continuous linear operators with finite range.

[Taam, [60], p. 41]

Theorem 0.6 : A completely continuous linear operator on a Banach space X to a Hilbert space Y can be approximated arbitrarily in norm by a sequence of continuous linear operators with finite range.

[Hildebrandt, [21] p.107]

Now we give a theorem 0.7 below which involves the notion of " Condition of approximation" , therefore, we first enumerate the equivalent conditions of approximation as follows, we take X and Y to be locally convex spaces.

(A) <u>Condition of approximation</u> : The identity operator on X into itself is adherent to the tensor product, $X^* \otimes X$ in $L_C(X, X)$. In other words for every precompact subset $U \subseteq X$ and every neighbourhood V of 0 in X, there exists a continuous endomorphism T, with range Tx such that

$$Tx - x \in V, \quad \text{for every } x \in U .$$

(A_1) $X^* \otimes X$ is dense in $L_c(X,X)$ i.e. every continuous linear operator on X into itself can be approximated uniformly on every precompact subset by continuous linear operators with finite range.

(A_2) $X^* \otimes Y$ is dense in $L_c(X,Y)$

(A_3) $Y^* \otimes X$ is dense in $L_c(X,Y)$

Note : : The above conditions A, A_1, A_2 and A_3 are equivalent.

[Grothendick,[55] proposition 35pp164-165]

Theorem 0.7 : In order that every completely continuous linear operator on X to Y be the limit in $L_c(X,Y)$ of continuous linear operators with finite range, it is sufficient that Y or the dual of X, X^* satisfies the condition of approximation.

[Grothendick, [55] p. 168]

Theorem 0.8 : Let $\mathfrak{F}$ be a pointwise family of continuous linear operators (continuous semi-norms) on a Banach space X to a Banach space Y. Then $\mathfrak{F}$ is uniformly bounded.

[Wilansky, [64], p. 116]

Theorem 0.9 : Let X,Y be linear topological spaces (F-spaces) and suppose that $\{T_n\}$ is a sequence of continuous linear operators on X to Y such that $T(x) = \lim T_n(x)$ exists for all x in a subset U of X of the second category. Then T is continuous and the convergence is uniform on every totally bounded set.

[Kelley and others, [63] p.105]

Remark 0.12 : If X is complete space in the statement of Theorem 0.9 then the "uniform convergence on a totally bounded set" in the statement of the theorem is replaced by uniform convergence on a relatively compact set.

Theorem 0.10 : Let $\{p_n\}$ be a sequence of paranorms (semi-norms) on a linear space X and $\{x_n\}$ be a sequence in X. Let

$$p(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1 + p_k(x)}$$

be Frechet combination of $\{p_n\}$ Then (i) p is a paranorm and satisfies the condition $p(x_n) \to 0$ iff $p_k(x_n) \to 0$ for each k,

(ii) p is the weakest paranorm which is stronger than every p_k ,

(iii) p is total if and only if $\{p_k\}$ is a total set

[Wilansky, [64] p. 54]

Theorem 0.11 : Let U be a set in a locally convex space X with $\sigma(\Phi)$ the topology determined by family Φ of semi-norms. Then U is bounded if and only if p(U) is bounded for each $p \in \Phi$.

[Wilansky, [64] p. 216]

Theorem 0.12 : Every non-normable F-space X is homeomorphic with all of its closed convex bodies.

[Bessaga and Klee [66] p. 161]

Theorem 0.13 : A closed, connected, locally convex subset U of a Hausdorff linear topological space X is convex.

[Kelley and others, [63],p.48]

Theorem 0.14 : Suppose U is a closed linear subspace of finite co-dimension m in a non-normable F-space X, and V is a closed convex body in U. Then V is homeomorphic with X.

[Bessage and Klee [60], p. 164]

Theorem 0.15 : Let X be a linear space and Φ a total collection of linear functionals on X. Then the weak topology $w(X, \Phi)$ is metrizable if and only if Φ is of countable Hamel dimension in $X^{\#}$, where $X^{\#}$ is the set of linear functionals on X. It is normable if and only if Φ is of finite dimension.

[Wilansky, [64], p. 150]

Theorem 0.16 : Let Φ be a total family of semi-norms on a linear space X. Then if Φ is countable, $\sigma(\Phi)$ is metrizable. Conversely if $\sigma(\Phi)$ is metrizable, Φ is reducible to a countable set. Further Φ is finite, $\sigma(\Phi)$ is normable. Conversely if $\sigma(\Phi)$ is normable, Φ is reducible to a finite set.

[Wilansky, [64], p.217]

Remark 0.13: Any sequence of semi-norms can be replaced by an increasing sequence which determines the same linear topology.

[Wilansky, [64],problem 7,p.218]

Theorem 0.17 : Let T be a continuous linear operator on a Banach space X to a reflexive Banach space Y. If T maps weakly convergent sequences into normed convergent sequences, then T is strictly singular.

[Whitely, [64], p. 253]

Theorem 0.18 : Let T be a continuous linear operator on a Banach space X to an almost reflexive Banach space Y and transforms weakly converging sequence into strongly converging one, then T is strictly singular.

[Lacey and Whitely [65] p. 2]

Theorem 0.19 : If U is a closed subspace of a locally convex space X with the topology $\sigma(\Phi)$, the topology generated by a family of semi-norms defined on X and U^o is its polar in X^*, then X^*/U^o with quotient topology derived from the weak topology for X^* generated by a family of semi-norms defined on X^*, is topologically isomorphic to U^* with the weak locally convex topology generated by a family of semi-norms defined on U^*.

[Kelley and others [63] pp.160-161]

Theorem 0.20: Let T be a strictly singular operator on a reflexive Banach space X to a Banach space Y and X^* is subprojective space then T^* is strictly singular on Y^* to X^*.

[Whitely, [64], p. 254]

Theorem 0.21 : Let T be a continuous linear operator on a Banach space X to subprojective Banach space Y and T^* be a strictly singular operator on Y^* to X^*, then T is strictly singular.

[Whitely, [64], p. 254]

Theorem 0.22 : Let X be a finite dimensional inner product space and let T be a projection operator. The following statements are equivalent.

(i) T is normal

(ii) T is self-adjoint

(iii) T is the orthogonal projection operator on its range.

[Bachman, [66], p. 26]

Theorem 0.23 : If T is a continuous linear operator on a complex Hilbert space X to a complex Hilbert space Y, then,

$$\overline{R(T)} = R(T) = N(T^*)$$

where $\overline{R(T)}$ denotes the closure of orthogonal complement of the range of T, and $N(T^*)$ denotes the null space of T^*.

[Bachman, [66], p.363]

Theorem 0.24 : If T is a normal operator on a Hilbert space X, $T : X \to X$, then,

$$\pi(T) = \sigma(T)$$

[Bachman, [66], p. 380]

Theorem 0.25 : If T is a completely continuous and normal operator on a Hilbert space X, then $P_\sigma(T) \neq 0$, and there is some $\lambda \in P_\sigma(T)$ such that $|\lambda| = \|T\|$.

[Bachman, [66], p.386]

Theorem 0.26 : Every completely continuous hyponormal operator is normal.

[Ando, [63] p.291]

Theorem 0.27 : Suppose T is completely continuous linear operator on a complex normed space X viz. $T : X \to X$. Then $P_\sigma(T)$ is almost countable (it could be empty) and 0 is its possible limit point.

[Bachman, [66], p. 299]

Theorem 0.28 : Let T be a completely continuous linear operator on a normed linear space X to a normed linear space Y, R(T), the range of T is seperable.

[Bachman, [66] p. 291]

Theorem 0.29 : A Hilbert space X has countable (finite) dimmension if and only if it is seperable.

[Wilansky, [64],p.130]

Theorem 0.30 : Let X be a complex finite dimensional inner product space and let T be a normal operator on X, $T : X \to X$, then there exists an orthonormal basis for X consisting of eigen vectors of T.

[Bachman [66], p. 24]

Theorem 0.31 (i) If $\{f_n(z)\}$ is a sequence of analytic functions, regular in a closed domain D bounded by a closed contour C and if $f_n(z)$ converges uniformly to f(z) in D (f(z) does not vanish on C), then f(z) and the functions $f_n(z)$, for sufficiently large values of n, all have the same number of zeros within C.

(ii) Prove also a zero of f(z) is either a zero of $f_n(z)$ for sufficiently large values of n or else is a limiting point of the zeros of the functions of the sequence.

[Copson [35], p. 121]

Theorem 0.32 : If f(x,y,t) is a function of three variables, defined in three dimensional space which is continuous with respect to x,y,t seperately then f(x,y,t) is atmost a pointwise discontinuous function.

[Hobson [27], p. 450]

Theorem 0.33 : Let $\{f_n(z)\}$ be a sequence of functions, each regular in a region D, let , $|f_n(z)| \leq K$, for all n and z in D, K being a positive constant and let $f_n(z)$ converges to a limit f(z) at everywhere dense set of points in D. Then $f_n(z)$ converges to f(z) uniformly in any region bounded by a contour interior to D, the limit f(z) being, therefore, an analytic function of z.

[Titchmarsh [32] p. 168].

REFERENCES

Riesz and Nagy [55], Maddaus [38], Phillips [40], Taam [60], Hilderhandt [31], Grothendick [5.], Wilansky [64], Kelley and others [63], Bossage and Klee [66], Whitely [64], Lavy and Whitely [65],. Bachman [66], Ando [63], Copson [35], Hobson [27], Titchmarsh [32].

CHAPTER - I

COMPLETELY CONTINUOUS LINEAR OPERATOR

1:1 Introduction :

In this chapter we deal with well known open problem [Problem 0.1] stated as : Let T be a completely continuous linear operator on a Banach space X to a Banach space Y. Can T be approximated arbitrarily close in norm by a sequence of continuous linear operators with finite range ? [Hilla and Phillips [87] p.49] Before 1931, it was shown that the answer is in the affirmative when (i) both X and Y are Hilbert spaces [Theorem 0.2]. In 1931, T. Hilderhandt proved that the answer is in the affirmative when there is no restriction on X but Y is a Hilbert space [Theorem 0.6]. In 1938, Ingo Maddaus proved that the answer is in the affirmative when there is no restriction on X, but Y is of "type A" [Theorem 0.3]. In 1940, R.S.Phillips proved that the answer is in the affirmative when X is either L^p or C and there is no other restriction on Y [Theorem 0.4]. In 1955, Alexandie Grothendick proved that the answer is in the affirmative when X and Y are locally convex spaces but X^* or Y satisfies the " Condition of Approximation". [Theorem 0.7] In 1960, C.T.Taam proved that the answer is in the affirmative when there is no other restriction on X but Y is C(E) [Theorem 0.5]. In the last we study the effect of hyponormality on the known result [Theorem 0.22] which is true for normality and obtain the conditions under which a complex Hilbert space X has an orthonormal Schauder basis consisting of eigen vectors of the operator on the space X.

The condition of approximation is stronger condition than that of " type A". We extend the definition of " type A" to F-spaces and prove with weaker hypothesis that the answer is also in the affirmative for the following types of domains and ranges. Our method of proof is quite different from that of Theorem 0.7 :

(i) X is a F-space and Y is a Banach space of " type A "

(ii) X is a Banach space and Y is a F-space of " type A"

(iii) X is a F-space and Y is a F-space of " type A"

1.2 Study of the problem on a F-space to a Banach space of " type A".

For the study of the problem when X is a F-space and Y is a Banach space of " type A", we give some definitions and prove a lemma and obtain the main result.

Definition 1.1 : Let T be a continuous linear operator on a F-space X to a Banach space Y, and $\{p_n\}$ be a sequence of semi-norms on X which defines the topology on X.

We define the norm as

$$\sup_{p_n(x)\leq 1} \| Tx \| = \| T \|_n$$

Definition 1.2 : Let T_m be a sequence of continuous linear operators and T be a continuous linear operator on a F-space X to a Banach space Y of " type A" . We say that T_m converges to T uniformly if $\| T_m - T \|_n$ converges to zero for each n = 1,2,3,.

Now we state the lemma :

Lemma 1.1 : Let Y be a Banach space of " type A" [Def. 0.57] and X be a F-space.

$$\text{Let } T_r(x) = \sum_{s=1}^{r_s} L_{rs}(y)\, y_s ,$$

$$U_r(y) = y - \sum_{s=1}^{r_s} L_{rs}(y) y_s ,$$

and also let $Tx = y$, $Tx_s = y_s$ such that $p_n(x_s) \leq 1$. Then $\| U_r(y)\|$ converges to zero, uniformly on every relatively compact set V of Y.

Proof : Since obviously each $U_r(y)$ is finite dimensional and $\lim \| U_r(y)\| = 0$ for each $y \in Y$ because Y is a Banach space of type A [Definition 0.57], there exists a positive constant K independent of both y and r such that $\| U_r(y)\| < K \| y \|$ [Theorem 0.8]. V is relatively compact ==> V is totally bounded ==> if $\epsilon > 0$ arbitrarily chosen then there is a finite set $y_1, y_2, y_3, \ldots, y_p$ of V such that $y \in V$ is interior to at least one of the spheres of centres y_i $(i = 1,2,\ldots,p)$ and of radius ϵ/K. Obviously $\| U_r(y)\|$ approaches zero uniformly on the finite set y_i $(i = 1,2,3,\ldots,p)$ so that for $r > r_o(\epsilon)$ we have $\| U_r(y)\| \leq \epsilon$ on this set. Then for any $y \in V$ and some y_i

$$\big| \|U_r(y)\| - \| U_r(y_i)\| \big| \leq \| U_r(y - y_i)\| \leq K \| y - y_i\| < \epsilon$$

hence $\| U_r(y)\| < \epsilon$ for each $y \in V$ when $r > r_o(\epsilon)$. This proves the lemma.

Theorem 1.1 : Every completely continuous linear operator on a F-space X to a Banach space Y of " type A" is the uniform limit of a sequence of continuous linear operators with finite range.

Proof : Let T be a completely continuous linear operator on X to Y, then

$$T_r(x) = \sum_{s=1}^{r_s} L_{rs}(y)\, y_s$$

finite dimensional operator on X to the closed linear manifold generated by the finite number of elements y_s in Y ($s = 1,2,..r_s$) which forms a subspace of Y. Let V_1 be the transform by T of those elements U_1 of X for which $p_n(x) \leq 1$, $x \in U_1$. Since T is completely continuous, the set V_1 is relatively compact by lemma 1.1 and $\| U_r(y)\|$ converges to a zero uniformly on V_1. In other words $U_r(Tx) = Tx - T_r x = (T - T_r)x$

$$\sup_{p_n(x)\leq 1} \| U_r(Tx)\| = \sup_{p_n(x)\leq 1} \| (T - T_r)x \|$$

$$= \| T - T_r\|_n$$

converges to zero for each n(n = 1,2,3,...). This proves the theorem.

1.3 Study of the problem on a Banach space to a F-space of " type A".

In order to study the problem, when X is a Banach space and Y is a F-space of " type A", we extend the definition of

" type A" to F-space and give required definitions and prove two elemmas required for proving the main result.

Definition 1.3 : A F-space Y is said to be of "type A" if there exists a linearly independent sequence $\{y_s\}$ of elements of Y such that q_m is a sequence of seminorms on Y which defines the topology on Y satisfying $q_m(y_s) \leq 1$ for each y_s with a double sequence $L_{rs}(y)$ of continuous linear operators on Y such that for every $y \in Y$

$$\lim_{r\to\infty} q_m \left(y - \sum_{s=1}^{r_{s_m}} L_{rs}(y)\, y_s \right) = 0$$

Definition 1.4 : Let T be a continuous linear operator on a Banach X to a F-space Y of " type A" and q_m be a sequence of semi-norms which defines the topology on Y.

We define the norm $D_m(T)$ as

$$D_m(T) = \sup_{\|x\| \leq 1} q_m(Tx)$$

Definition 1.5 : Let $\{T_n\}$ be a sequence of continuous linear operators and T be a continuous linear operator on a Banach space X to a F-space Y of " type A". We say that $T_n \to T$ uniformly if $D_m(T_n - T) \to 0$ as $n \to \infty$, for each $m = 1,2,3,\ldots$.

Now we state the first lemma.

Lemma 1.2 Let X be a Banach space and Y be a F-space of " type A" [Definition 1.3]. Let

$$T_{rm}(x) = \sum_{s=1}^{r} {}^{s_m} L_{rs}(y)\, y_s$$

and

$$U_{rm}(y) = y - \sum_{x=1}^{r} {}^{s_m} L_{rs}(y)\, y_s$$

then $q_m(U_{rm}(y))$ converges uniformly to zero as $r \to \infty$ for $m = 1,2,3,\ldots$ on a relatively compact subset V of Y.

<u>Proof</u> : From Definition 1.3

$$\lim_{r \to \infty} q_m [\, U_{rm}(y)] = 0$$

for each $y \in Y$, for $m = 1,2,3,\ldots$ $\left\{U_{rm}(y)\right\}$ is the double sequence. of continuous linear operators on Y because each $L_{rs}(y)$ is continuous linear operator on Y. Y is a F-space ==> Y is complete ==> Y is of second category. By Theorem 0.9 and remark 0.12 it follows that $q_m [U_{rm}(y)]$ converges uniformly to zero as $r \to \infty$ for $m = 1,2,3,\ldots$ on a relatively compact subset V of Y. This completes the proof of the lemma.

We state the second lemma

Lemma 1.3 : Let Y be a F-space and q_m be a total sequence of semi-norms on Y which defines the topology by q on Y, given as

$$q(y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{q_m(y)}{1+q_m(y)}, \quad y \in Y$$

Then a set V in Y is compact in the topology induced by q if and only if it is compact in each of the topologies on Y induced by q_m , $m = 1,2,3,\ldots,$

Proof : Without any loss of generality, we can replace any sequence of semi-norm by a directed sequence of semi-norms which determines the same linear topology [Remark 0.13]. If V is compact in q, then, by theorem 0.10 (ii) and (iii), q is the weakest total paranorm which is stronger than total semi-norm and hence than total paranorm q_m $(m = 1,2,3,\ldots)$, therefore V is compact in each q_m ($m = 1,2,3,\ldots.$). Now we show that if V is compact in each $q_m(m = 1,2,3,\ldots)$ then V is compact in q . Let us consider a sequence $\{y_n^{(1)}\}$ in V. It contains a convergent subsequence $y_{11}^{(1)}, y_{12}^{(1)}, y_{13}^{(1)} \ldots y_{1n}^{(1)} \ldots$ converging to a limit y_1 of V in q_1 because V is compact in q_1.

Now $\{y_{1n}^{(1)}\}$ is sequence which has a convergent subsequence

$$y_{21}^{(2)}, y_{22}^{(2)}, y_{23}^{(2)} \quad ,\ldots, y_{2n}^{(2)} \ldots.$$

converging to y_2 of V in q_2 because V is compact in q_2. Since $\{q_m\}$ is a directed sequence, there is a q_k which is stronger than q_1 and q_2 . Then the sequence $\{y_{2n}^{(2)}\}$ contains a subsequence $u_1, u_2, u_3, \ldots, u_n \ldots$ which converges to a point u of V in q_k. Since the sequence $\{u_n\}$, being a subsequence of $\{y_{1n}^{(1)}\}$ and $\{y_{2n}^{(2)}\}$ converges to the points y_1 and y_2 respectively in the weaker total semi-norms (total paranorms) q_1 and q_2 and

$\{q_m\}$ is total, by uniqueness property of a limit of a sequence $\Rightarrow y_1 = y_2 = u$. Hence the subsequence extracted from this successively will converge to the same u of V. In general

$$y^{(m)}_{m_1},\ y^{(m)}_{m2},\ \ldots,\ y^{(m)}_{mm}\ \ldots.$$

will contain a subsequence converging to the same point u of V in each $q_m (m = 1,2,3,\ldots)$ By using the diagonal process, we can extract the sequence

$$y^{(1)}_{11},\ y^{(2)}_{22},\ \ldots,\ y^{(r)}_{rr}\ \ldots\ldots$$

which will converge to the same limit u in each of the semi-norms q_m $(m = 1,2,3,\ldots)$ The sequence

$$y^{(1)}_{11},\ y^{(2)}_{22},\ \ldots,\ y^{r}_{rr}\ \ldots.$$

being the single sequence can be written as

$$z_1,\ z_2,\ z_3,\ \ldots,\ z_n\ \ldots\ldots$$

By Theorem 0.10 (1) $z_n \to u$ in q. Hence V is compact in q. This proves the lemma.

Theorem 1.2 : Every completely continuous linear operator on a Banach space X to a F-space Y of " type A'', on which the topology is induced by a total paranorm q is defined by a total sequence $\{q_m\}$ of semi-norms on Y, can be approximated uniformly by a sequence of continuous linear operators with finite range on X to Y.

Proof : Let T be a completely continuous linear operator on X to Y and also let $Tx = y$, $x \in X$, $y \in Y$. Y is a F-space of " type A" ==>

$$T_{rm}(x) = \sum_{s=1}^{r_{s_m}} L_{rs}(y)\, y_s$$

is finite dimensional operator on X into the closed linear manifold generated by the finite number of elements y_s ($s = 1, 2, \ldots, r_{s_m}$) which forms a subspace of Y. Let V_1 be the image by T of those elements U_1 of X, satisfying $\|x\| \leq 1$ for any $x \in U_1$.

Since T is completely continuous, the set V_1 is relatively compact in Y, because by lemma 1.3 a set in Y is compact if and only if it is compact in each of the semi-norms q_m, ($m = 1, 2, 3, \ldots$) and by lemma 1.2

$$q_m \left(y - \sum_{s=1}^{r_{s_m}} L_{rs}(y)\, y_s\right) \to 0$$

uniformly on V_1 for each $m = 1, 2, 3, \ldots$

$$q_m(Tx - T_{rm}\, x) \to 0 \text{ as } r \to \infty$$

uniformly on V_1 for each $m = 1, 2, 3, 4, \ldots$

Thus

$$\sup_{\|x\| \leq 1} q_m(T - T_{rm})\, x = D_m(T - T_{rm}) \to 0 \qquad (1.a)$$

for each $m = 1, 2, 3, \ldots\ldots$

$\{T_{rm}\}$ is a double sequence of continuous linear operators with finite range, Using the diagonal process

$$\begin{array}{lllll} T_{11} & T_{21} & T_{31} & T_{41} & \cdots\cdots\cdots \\ T_{12} & T_{22} & T_{32} & T_{42} & \cdots\cdots\cdots \\ T_{13} & T_{23} & T_{33} & T_{43} & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots \end{array}$$

We choose the sequence containing diagonal elements

$$T_{11},\ T_{22},\ T_{33},\ T_{44},\ \ldots,\ T_{rr}\ \ldots\ldots$$

Therefore, (1.a) can be interpreted as

$$D_m(T - T_{rr}) \to 0$$

as $r \to \infty$, for each $m = 1,2,3,\ldots.$

Since the sequence $\{T_{rr}\}$ is a single sequence, it can be replaced by a single sequence T_k $\Longrightarrow$ $D_m(T - T_k) \to 0$ as $k \to \infty$ for each $m = 1,2,3,\ldots.$

This proves the theorem.

1.4 Study of the problem on a F-space X to a F-space Y "type A".

In order to study the problem, when X is a F-space and Y is a F-space of " type A". We give some definitions and then prove the main result .

Definition 1.6 : Let T be a continuous linear operator on a F-space X on which the topology induced by a total paranorm p as in [def. 0.76] is defined by a total sequence $\{p_n\}$ of semi-norms defined on X to a F-space Y of " type A " on which the topology induced by a total paranorm q as in [def.0.76] is defined by a total sequence q_m of semi-norms defined on Y.

Let $Tx = y$, $x \in X$, $y \in Y$. We define the norm $D_{mn}(T)$ as

$$D_{mn}(T) = \sup_{p_n(x) \leq 1} (Tx)$$

Definition 1.7 : Let $\{T_k\}$ be a sequence of continuous linear operators and T be continuous linear operator on a F-space X on which the topology induced by a total paranorm p is defined by a total sequence $\{p_n\}$ of semi-norms on X to a F-space Y on which the topology induced by a total paranorm q is defined by a total sequence $\{q_m\}$ of semi-norms on Y. We say that T_k converges uniformly to T if $D_m(T - T_k)$ converges to zero as $k \to \infty$ for $m = 1,2,3,\ldots\ldots$

Theorem 1.3 : Every completely continuous linear operator on a F-space X on which the topology induced by a total paranorm p is defined by a total sequence $\{p_n\}$ of semi norms on X to a F-space of " type A" on which the topology induced by a total paranorm q is defined by a total sequence $\{q_m\}$ of semi-norms on Y, can be approximated uniformly by a sequence of continuous linear operators with finite range on X to Y.

Proof : Let T be a completely continuous linear operator on X to Y and also let $Tx = y$, $x \in X$, $y \in Y$, Y is a F-space of '' type A'' [def. 1.3]

$$\Longrightarrow \quad T_{rm}(x) = \sum_{s=1}^{r_{s_m}} L_{rs}(y)\, y_s$$

is a finite dimensional operator on X into the closed linear manifold generated by the finite number of elements y_s ($s = 1,2,3,\ldots, r_{s_m}$) which form a subspace of Y. Let V be the image by T of those elements U of X, satisfying $p_n(x) \leq 1$, $x \in U$. U is bounded in X by Theorem 0.11. Since T is completely continuous by lemma 1.3 the set V is relatively compact in Y., as a set in Y is compact if and only if it is compact in each of the semi-norms q_m ($m = 1,2,3,\ldots$) Also, by lemma 1.2

$$q_m\Big(y - \sum_{s=1}^{r_{s_m}} L_{rs}(y)\, y_s\Big) \to 0$$

uniformly on V for each $m = 1,2,3,\ldots$.

Thus

$$\sup_{p_n(x) \leq 1} q_m(T - T_{rm})x = D_{mn}(T - T_{rm}) \to 0 \qquad (1.b)$$

for each $m,n = 1,2,3,\ldots$.

Using the diagonal process as in the proof of Theorem 1.2, (1.b) can be interpreted as

$$D_{mn}(T - T_k) \to 0 \qquad \text{as } k \to \infty$$

for each $m,n = 1,2,3,\ldots$

This completes the proof of the theorem.

1.5 Equivalent conditions of Hyponormality :

We obtain equivalent conditions of hyponormality if T is replaced by hyponormality and X is infinite dimensional in the statement of Theorem 0.22, prove the following theorem if T is hyponormal and X is a Hilbert space in the statement of Theorem 0.22 and prove three corollaries which are direct consequences of the theorem.

Theorem 1.4 : Let T be a projection operator on a complex Hilbert space X, then the following statements are equivalent:

(i) T is hyponormal

(ii) T is self-adjoint

(iii) T is the orthogonal projection operator on its range.

Proof : (i) ===> (ii)

T is hyponormal ==> $\| T^* x \| \leq \| Tx \|$ for all $x \in X$ from which it is obvious if $Tx = 0$, then $T^*x = 0$.

Consider the vector $z = Tx - x$, $x \in X$

$$Tz = T(Tx) - Tx = Tx - Tx$$

(because T is a projection operator)

$$==> Tz = 0 ==> T^*z = 0$$

Also we have

$$0 = T^*z = T^* Tx - T^*x$$

$$===> T^* Tx = T^*x ==> T^* T = T^* \qquad (1.c)$$

Computing the adjoint of each side

$$(T^* T)^* = (T^*)^*$$

or

$$T^* T^{**} = T^{**}$$

$$\Longrightarrow \quad T^* T = T \qquad (1.d)$$

(because T is closed)

From (1.c) and (1.d) we have $T = T^*$ which proves (i) ==> (ii).

Proof:(ii) ==> (iii)

Following the proof of Theorem 0.23 [Bachman [66] p. 363] we obtain

$$R(T)^{\perp} = N(T^*)$$

But

$$T = T^*$$

Therefore

$$R(T)^{\perp} = N(T)$$

==> T is orthogonal projection operator .

Proof (iii) ---> (i)

Let $x, y \in X$ be arbitrary. Since T being projection operator, is idempotent ==> $x - Tx \in N(T) = R(T)^{\perp}$. And $Ty \in R(T)$. Hence $(x - Tx, Ty) = 0$ ==> $(x, Ty) - (Tx, Ty) = 0$.

Thus ,

$$(x,Ty) = (Tx, Ty)$$

$$= (x, T^* Ty) \qquad \text{from Def. 0.67.}$$

Therefore $T = T^* T$ and then following the proof as in proving (i) ==> (ii) we get $T = T^*$ ==> T is self-adjoint ==> T is normal which is certainly hyponormal. This completes the proof of the theorem.

Corollary 1.1 : Let T be a projection hyponormal operator on a Hilbert space X, then $w(T) = \sigma(T)$ and the entire spectrum is real.

Proof: It immediately follows from Theorem 1.4 and Theorem 0.24.

Corollary 1.2 : Let T be a completely continuous hyponormal projection operator on a Hilbert space X, then the point spectrum $P_\sigma(T) \neq 0$, and there is some $\lambda \in P_\sigma(T)$ such that $|\lambda| = \|T\|$.

Proof : The proof follows easily from Theorem 1.4 and theorem 0.25.

Corollary 1.3 : Let T be a completely continuous hyponormal operator on a Hilbert space X, then $P(T) \neq 0$, and there is some $\lambda \in P_\sigma(T)$ such that $|\lambda| = \|T\|$.

Proof : T is completely continuous hyponormal, by Theorem 0.26, T is normal and the proof follows from Theorem 0.25.

1.6 Hyponormality and Orthonormal Schauder basis :

In this section we study the conditions under which a complex Hilbert space has an orthonormal Schauder basis consisting of eigen vectors of an operator on the space. We state and prove the following Theorem.

Theorem 1.6 : Let X be a complex Hilbert space and let T be a completely continuous hyponormal operator on X such that $T(X) = X$. Then there exists an orthonormal Schauder basis for X consisting of eigen vectors of T.

Proof : Since T is completely continuous and hyponormal, by Corollary 1.3, T must have eigen values. Since T is completely continuous operator on X, by Theorem 0.27 the point spectrum $P_{\sigma}(T)$ is countable. Let $x_1, x_2, x_3, \ldots, x_n \ldots$ be eigen vectors of T associated with the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n \ldots$ such that $\| x_n \| = 1$ for all n (This is always possible because any scalar multiple of an eigen vector is still an eigen vector). Again, by theorem 0.28, $T(X)$, the range of T is seperable. But $T(X) = X$, by hypothesis, therefore X is seperable. Now X is a seperable Hilbert space. Thus, by Theorem 0.29, $x_1, x_2, x_3, \ldots, x_n \ldots$ is a Schauder basis for X. This completes the theorem.

Remark 1.1 : It may be noted that Theorem 1.6 extends Theorem 0.30 to Hilbert spaces.

REFERENCESS

Hilla and Phillips [57], Riesz and Nagy [55], Hildebrandt [31], Maddaus [38], Phillips [40], Grothendick [55], Taam [60], Wilansky [64], Bachman [66].

CHAPTER - II

SUPER OPERATOR AND ITS EXISTENCE

2.1 Introduction :

In this chapter we introduce the concept of Super operator on F-spaces. We shall also call it S-operator. We develop the theory of S-operator in this chapter as (1) Relation between S-operators and strictly singular operators (2) Existence theorems.

2.2 Concept of S-Operator :

In 1958, T.Kato introduced the concept of a strictly singular operator on Banach spaces [Def. 0.64] and developed its theory Kato, [58] (pp. 261-332) . The concept of a strictly singular operator extends that of a completely continuous operator because a normed space X is finite dimensional if and only if a closed and bounded set in X is compact. We introduce in the following a new operator.

Definition 2.1 Super Operator (S-operator) : Let T be a continuous linear operator on an F-space X to an F-space Y. We call T a Super Operator if whenever the restriction of T to a closed subspace U of X has a continuous inverse, U is a Montel space. By the existence of continuous inverse we shall mean that the restriction of T to a closed subspace U of X is a homeomorphism.

Remark 2.1 : It is easy to see that the sum, product and the quotient of two-S-operators need not be S-operator because Montel space involved in Def. 2.1 may be infinite dimensional. We may however develop some algebra by use of transfinite arithematic but we do not attempt here any thing in that direction.

Remark 2.2 : Again the limit of a sequence of S-operators need not be S-operator for the same reasons as in Remark 2.1.

2.3 Relation between S-operators and strictly singular operators:

We remark in this section how we got the concept of an S-operator from the concept of a strictly singular operator and relate them together. The concept of strictly singular induced the concept of S-operator. This follows from the fact that the existence of infinite dimensional Montel spaces together with the fact that a normed space is a Montel space if and only if it is finite dimensional (Kelley [63] p. 196).

The S-operator therefore becomes strictly singular operator on Banach spaces and thus the concept of an S-operator on F-spaces extends that of a strictly singular operator on Banach spaces. We state and prove the following theorem relating S-operator and strictly singular operator.

Theorem 2.1 : Every strictly singular operator on a non-normable F-space with the topology induced by a total paranorm p on X (defined by a total sequence $\{p_n\}$ of semi-norms on X) to a non-normable F-space with the topology induced by a total paranorm q (defined by a total sequence $\{q_m\}$ of semi-norms on Y) is an S-operator, but the converse is not true.

Proof : Let T be a strictly singular operator on X to Y ==> The restriction of T to a closed subspace U of X is a homeomorphism, U is finite dimensional [Def. 0.64] ==> U is a Montel space ==> T is S-operator [Def. 2.1]. The existence of infinite dimensional Montel spaces shows that the converse is not true. This completes the proof of the theorem.

2.4 Existence theorems of an S-operator :

We state and prove four results on existence of an S-operator on F-spaces.

Throughout this section we shall take X,Y as non-normable F-spaces on which the topologies induced by the total paranorms p and q defined by total sequences $\{p_n\}$ and $\{q_m\}$ of semi-norms respectively as in [Def. 0.76].

Theorem 2.2 (First Existence Theorem): Every continuous linear operator on a non-normable F-space X with the topology induced by the total paranorm p on X to the space of all sequences <s> with the metric

$$d(x,0) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1+|x_i|}, \quad \text{where } \{x_i\} \in \langle s \rangle$$

is S-operator.

Proof : Let T be a continuous linear operator on X to <s> , U be a closed subspace of X and the restriction of T to U is a homeomorphism. In order to show that T is S-operator, we have to show U is Montel space. For showing that U is a Montel space

We have to show that a closed subspace of <s> being a Montel space, is also a Montel space because homeomorphic image of a Montel space is a Montel space. Since U is a closed subspace of X, the restriction of T to U is a homeomorphism, T(U) is a closed subspace of <s> ==> T(U) is connected. Also <s> is an F-space and therefore it is locally convex and complete ==> T(U) is locally convex and complete. Since T(U) is closed, connected, locally convex linear metric space with metric d-restriction to T(U), hence, by theorem 0.13, T(U) is convex. T(U) being a closed subspace of a complete space <s>, is complete ==> T(U) has a non-empty interior ==> T(U) is a closed convex body in <s> . By Theorem 0.12, <s> is homeomorphic with T(U), but <s> is an infinite dimensional Montel space [Pelczynski and Rolewicz, [59] pp. 45-51], therefore T(U) is infinite dimensional Montel space. Since T(U) is the homeomorphic image of U, U is an infinite dimensional Montel space. Hence T is an S-operator. This completes the theorem.

Remark 2.1 : In the proof of the theorem it is proved that U and T(U) are infinite dimensional subspaces of X and <s>, that is to say, U and T(U) are of finite codimension in X and <s> respectively.

Remark 2.2 : T is an S-operator in Theorem 2.2.but from remark 2.1, it is not a strictly singular operator.
Theorem 2.3 (Second Existence Theorem) : Every continuous linear operator on a non-normable F-space X with the topology induced by the total paranorm p on X to the space of complex valued functions A(U) as in [Def. 0.77], is an S-operator.

Proof : A(U) with the metric [Def. 0.77] is a non-normable F-space which is also a Montel space (Kelley [63] pp.196-197) The remaining part of the proof follows easily from the same argument as in Theorem 2.2.

Theorem 2.4 (Third Existence Theorem) Every continuous linear operator on a non-normable F-space X with the topology induced by the total paranorm p on X to the space $\mathcal{D}$ of infinitely differential functions on R^n, is an S-operator.

Proof: $\mathcal{D}$ with the metric [Def. 0.78] is a non-normable F-space which is also a Montel space (Kelley, [63] p. 82 and p.197). The remaining part of the proof follows easily from the argument as that of Theorem 2.2. The above theorems suggest the following more general theorem.

Theorem 2.5 (Fourth Existence Theorem): Every continuous linear operator on a non-normable F-space X with the topology induced by the total paranorm p on X to a non-normable F-space Y which is also a Montel space with the topology induced by the total paranorm q on Y, is an S-operator.

Proof : The proof can be developed just as in Theorem 2.2.

REFERENCES

Kato [58, Kelley [63], Bessaga and Klee [66], Pelczynski and Rolewicz [59].

CHAPTER - III

SEPERABILITY OF AN S-OPERATOR

3.1 Introduction :

In this chapter we deal with the seperability of the range of an S-operator. It is well known that the range, R(T), of a completely continuous linear T on a normed space X to a normed space Y is seperable (Theorem 0.28). In 1963, Goldberg and Thorp in (Goldberg and Thorp [63] pp. 335-336) showed that the range of a strictly singular operator need not be seperable by an example which is given below.

If Q is an arbitrary set, $_p(Q)$, $1 < p < \infty$ is the space of scalar valued functions x with domain Q, having atmost countably non-zero coordinates such that

$$\| x \| = \left(\sum_{q \in Q} |x(q)|^p \right)^{\frac{1}{p}} \text{ is finite.}$$

It is a Banach space with this norm. It is asserted that all the continuous operators $T : \ell_2(Q) \to \ell_p(Q)$ ($2 < p < \infty$) where Q is an uncountable set, are strictly singular and that the inclusion map is such an operator and has a non-seperable range. From this it is obvious that the range of an S-operator on a normable F-space X to a normable F-space Y, is not seperable because an S-operator becomes strictly singular operator on Banach spaces (normable F-spaces). We prove in the next section that the range of an S-operator on a non-normable F-space X to a non-normable F-space Y is seperable.

3.2 Seperability of an S-operator :

We study the seperability of an S-operator on a non-normable F-space X with the topology induced by the total paranorm p on X as in[Def. 0.76] to a non-normable F-space Y with the topology induced by the total paranorm q on Y as in [Def. 0.76].

Now we state and prove the following theorem :

Theorem 3.1 : The range of an S-operator on a non-normable F-space X with the topology induced by the total paranorm p on X to a non-normable F-space Y with the topology induced by the total paranorm q on Y, is seperable.

Proof : Let T be an S-operator on X to Y ==> the restriction of T to a closed subspace U of X is a homeomorphism, U is a Montel space . Now T(U) is the image of U under T is the homeomorphic image of the Montel space U ==> T(U) is a Montel space. Since U is a closed subspace of X, T(U) being the homeomorphic image of a closed subspace U of X, is a closed subspace of Y. Let V be a closed and bounded set (sub-space) of T(U). From Theorem 0.11 V is bounded in T(U) or in q restriction to T(U). Since T(U) is Montel space and by lemma 1.3, a bounded and closed set in T(U) or in q restriction to T(U) is compact if and only if it is compact in each $q_m(m = 1,2,3,..)$ restriction to T(U), V is compact in T(U). We know that every compact metric space is complete and seperable (Kelley [63] p. 32) therefore V is seperable. From Remark 2.1, T(U) being a closed subspace of Y, is of finite codimension. It is easy to see as in the proof of Theorem 2.2, V is a closed convex body in T(U). By theorem 0.14, V is homeomorphic with Y, therefore Y is seperable. T(X) is a closed subspace of Y and so T(X) is seperable because every closed subspace of a seperable metric space is seperable.

Hence the range of T is seperable.

REFERENCES

Bachman [66], Goldberg and Thorp [63], Wilansky [64], Kelley [63], Bessage and Klee [66].

CHAPTER - IV

S-OPERATOR AND ALMOST REFLEXIVITY

4.1 Introduction :

In 1964, R.J.Whitely obtained [Theorem 0.17] a relation between strictly singular operator and reflexivity of Banach space. In 1965 Lacey and Whitely modified and obtained [Theorem 0.18] a relation between strictly singular operator and almost reflexivity of Banach space.

In this chapter, we study the relation between
(i) S-operator and reflexivity of F-space and also between
(ii) S-operator and Almost reflexivity of F-space.

4.2 Relation between S-operator and reflexivity of F-space :

We shall take X,Y as non-normable F-spaces with the topologies induced by total paranorms p,q on X and Y respectively defined by total sequences $\{p_n\}$ and $\{q_m\}$ of semi-norms on X and Y respectively as in [Def. 0.76].

We prove the following theorem relating S-operator and the reflexivity of F-space.

Theorem 4.1 : Let T be a continuous linear operator on a non-normable F-space with the topology induced by the total paranorm p on X to a non-normable reflexive F-space with the topology induced by the paranorm q on Y, and let T transform weakly converging sequence into strongly converging one, then T is an S-operator.

Proof : Suppose T_1 the restriction of T to a closed subspace U of X is a homeomorphism. Since U is homeomorphic to a closed subspace of the reflexive F-space Y and a closed subspace of a reflexive F-space is reflexive, U is reflexive. This implies that a closed and bounded set U_1 in U is weakly sequentially compact in the weak topology $w(X,\Phi)$ restriction to U. By Theorem 0.15, $w(X, \Phi)$ is metrizable and so is $w(X,\Phi)$ the restriction to U. By hypothesis and lemma 1.3, it follows that $T_1(U_1)$ is strongly sequentially compact in the semi-norm topology q restriction to $T_1(U)$. Since T_1 is homeomorphism, U_1 is strongly sequentially compact in the semi-norm topology p restriction to U. By Theorem 0.16, $\sigma(P_n)$ or p is metrizable so is $\sigma(P_n)$ restriction to U or p restriction to U is metrizable. We know that all the notions of compactness in linear metric space are equivalent ==> U_1 is compact, consequently U is a Montel space. Hence T is S-operator. This completes the proof of the theorem.

Remark 4.1 : Our Theorem 4.1 extends the result of Whitely [Whitely [64] p. 253] Theorem 0.17 to F-spaces.

4.3 Relation between S-operator and Almost reflexivity of F-space :

We shall denote X and Y as before as non-normable F-spaces with the topologies induced by total paranorms p,q on X and Y respectively, defined by total sequences $\{p_n\}$ and $\{q_m\}$ of semi-norms on X and Y respectively. We prove the following theorem relating S-operator and almost reflexivity of F-space.

Theorem 4.2 : Let T be a continuous linear operator on a non-normable F-space X with the topology induced by the total paranorm p on X, to a non-normable almost reflexive F-space Y with the topology induced by the total paranorm q on Y, and let T transform weakly converging sequence into strongly converging one, then T is an S-operator.

Proof : Suppose the restriction of T to a closed subspace U of X is a homeomorphism. Since U is homeomorphic to a closed subspace of the almost reflexive F-space Y and a closed subspace of an almost reflexive F-space is almost reflexive, U is almost reflexive ==> a bounded sequence in a closed and bounded set U_1 in U is weak cauchy sequence $\{x_n\}$. Now $\{x_n\}$ is a weak Cauchy sequence if and only if $[Tx_{n_{i+1}} - Tx_{n_i}]$ converges strongly to zero for each subsequence $\{x_{n_i}\}$ because by hypothesis, T transforms weakly converging sequence in X into strongly converging sequence in Y. This is true if and only if $\{Tx_n\}$ is a strong Cauchy sequence in $T(U_1)$ [Cauchy sequence in the semi-norm topology q restriction to $T(U)$]. Since $T(U)$ is the closed subspace of the F-space Y, $T(U)$ is complete therefore $\{Tx_n\}$ converges strongly in $T(U_1)$. T restriction to U is a homeomorphism therefore $\{x_n\}$ converges strongly in U_1. Since by lemma 1.3 a set in U or in p restriction to U is compact if and only if it is compact in each p_n restriction to U, U_1 is strongly sequentially compact in U or in p restriction to U. By Theorem 0.16, $\sigma\{p_n\}$ or p is metrizable so is $\sigma\{p_n\}$ restriction to U or p restriction to U is metrizable. We know that all the notions of compactness in a linear metric space are equivalent $\Longrightarrow$ U_1 is compact consequently U is a Montel space. Hence T is S-operator. This completes the proof of the theorem.

Remark 4.2 : Our Theorem 4.2 extends Lacey and Whitely theorem [Theorem 0.18] to F-spaces . For proof we have proceeded just as in the proof of Theorem 4.1 .

REFERENCES

Whitely [64], Lacey and Whitely [65], Wilansky [64].

CHAPTER - V

S-OPERATORS AND THEIR CONJUGATES

5.1 Introduction :

In 1963, Goldberg and Thorp showed that a conjugate operator of a strictly singular operator need not be a strictly singular operator. This has been verified if T is taken to be a continuous linear operator from a summable sequence, ℓ_1 onto a squared summable sequence, ℓ_2. Then T is strictly singular but T^* is not.

Also in 1964, Whitely showed that a continuous linear operator is not a strictly singular operator but its conjugate is a strictly singular operator. The example which he has chosen to assert the fact is given below :

Let T be an isometric isomorphism of the space of squared summable sequences, ℓ_2 into C[0,1], the space of continuous functions defined on [0,1]. T is not strictly singular but T^* is an operator from $C[0,1]^*$ into a reflexive space $(\ell_2)^*$ is strictly singular.

In 1964 Whitely proved under certain conditions :

(i) that the strict singularity of a continuous linear operator implies the strict singularity of its conjugate [Theorem 0.20].

(ii) also the strict singularity of a conjugate operator implies the strict singularity of the operator itself [Theorem 0.21].

On the same line we develop the following section.

We first define N-space :

Definition 5.1 N-Space : A non normable F-space X is said to be an N-space if for every subspace U of X, U^* is a closed subspace of X^* .

5.2 Relation between S-operator and its conjugate :

Let T be a continuous linear operator on an N-space with the topology induced by the total paranorm p on X, defined by a total sequence $\{p_n\}$ of semi-norms to a non-normable F-space Y with the topology induced by the total paranorm q on Y, defined by a total sequence $\{q_m\}$ of semi-norms on Y, and T^* be a continuous linear operator on a non-normable F-space Y^* with the topology induced by the metric d_2 as in [Def. 0.84] to a non-normable F-space X^* with the topology induced by the metric d_1 as in [Def. 0.84]. We shall first prove that if T is an S-operator, then T^* is an S-operator.

Theorem 5.1 : Let T be an S-operator on an N-space X with the topology induced by the total paranorm p on X to a non-normable F-space Y with the topology induced by the total paranorm q on Y and T^* be a continuous linear operator on a non-normable F-space Y^* with the topology induced by the metric d_2 to anon-normable F-space X^* with the topology induced by the metric d_1. Then T^* is an S-operator.

Proof : T is S-operator ==> if the restriction of T to a closed subspace U of X is homeomorphism, U is a Montel space. To show that T^* is an S-operator, supposing that the restriction of T^* to a closed subspace R of Y^* is a homeomorphism, we shall show that R is a Montel space. Now since

$$T : X \to Y \quad \text{and} \quad T^* : Y^* \to X^*$$

such that $U \to T(U)$ and $R \to T^*(R)$ it is easy to see that U is a closed convex body in X as we have seen in the proof of Theorem 2.2 and by Theorem 0.12, X is homeomorphic with U; also by remark 2.1, U is infinite dimensional which implies U is of finite co-dimension in X. Again U^* is a closed subspace of a non-normable F-space X^*, and hence U^* is a closed convex body in X^* as can be seen in the proof of Theorem 2.2. Similarly, by theorem 0.12 X^* is homeomorphic with U^* and $T^*(R)$ is a closed convex body in X^*; also by theorem 0.12, X^* is homeomorphic with $T^*(R)$. Thus X^* is homeomorphic with U^* and X^* is homeomorphic with $T^*(R)$ $\Longrightarrow$ U^* is homeomorphic with $T^*(R)$. But U^* is a Montel space with topology induced by the restriction of the metric d_1 to U^* since U is a Montel space and the strong dual of a Montel space is also a Montel space $\Longrightarrow$ $T^*(R)$ is a Montel space. Since $T^*(R)$ is a homeomorphic image of R, R is a Montel space. Hence T^* is an S-operator. Hence the result.

Remark 5.1 : Theorem 5.1 extends Whitely Theorems [Theorem 0.20] to F-spaces.

Let T be a continuous linear operator on a non-normable F-space X with the topology is induced by the total paranorm p on X, to a non-normable F-space Y on which the topology is induced by the total paranorm q on Y, T^* be a continuous linear operator on an N-space Y^* with the topology induced by the metric

d_2 as to a non-normable F-space X^* with the topology induced by the metric d_1 and T^{**} be a continuous linear operator on a non-normable F-space X^{**} with the topology induced by the total paranorm p restriction to X^{**} [Remark 0.11] to a non-normable F-space Y^{**} with the topology induced by the total paranorm q to restriction to Y^{**} [Remark 0.11]. We shall show that, if X is reflexive and T^* is an S-operator, then T is an S-operator.

Theorem 5.2 : Let T be a continuous linear operator on a non-normable F-space X with the topology induced by the total paranorm p on X, to a non-normable F-space Y with the topology induced by the total paranorm q on Y, T^* be an S-operator on an N-space Y^* with the topology induced by the metric d_2 to a non-normable F-space X^* with the topology induced by the metric d_1 and T^{**} be a continuous linear operator on a non-normable F-space X^{**} with the topology induced by the total paranorm p restriction to X^{**} to a non-normable F-space Y^{**} with the topology induced by the total paranorm q to restriction to Y^{**}. Then T is an S-operator.

Proof : Now since

$$T^* : Y^* \to X^*$$

and

$$T^{**} : X^{**} \to Y^{**}$$

such that

$$M_1 \to T^{**}(M_1)$$

Also

$$T : X \to Y$$

such that

$$M_1 \to T(M_1)$$

By theorem 5.1 and the assumption that T^* is an S-operator on

Y^* to X^*, T^{**} is an S-operator on X^{**} to Y^{**}. Therefore the restriction of T^{**} to a closed subspace M_1 of X^{**} which is equal to X as X is reflexive by hypothesis, is a homeomorphism and therefore M_1 is Montel space. Also M_1 being a closed subspace of a reflexive F-space X, M_1 is reflexive and $T^{**}(M_1)$ being the homeomorphic image of M_1, $T^{**}(M_1)$ is also reflexive. It is easy to see that $T^{**}(M_1) \subset Y^{**} \subset Y$ is a closed convex body in Y [proof of Theorem 2.2]. By Theorem 0.12, Y is homeomorphic with $T^{**}(M_1)$ and so Y is reflexive, that is, $Y = Y^{**}$ and by hypothesis X is reflexive, therefore $T = T^{**}$. Hence T is an S-operator on X to Y. This completes the proof of the theorem.

Remark 5.2 : Theorem 5.2 extends Whitely Theorem [Theorem 0.21] to F-spaces.

REFERENCES

Goldberg and Thorp [63], Whitely [64], Bessage and Klee [66].

CHAPTER - VI

SPECTRUM PROPERTY OF AN S - OPERATOR

6.1 Introduction :

In this chapter, we study the point spectrum property of S-operator on a non-normable F-space X. The theory of spectrum of S-operators may be developed but we do not attempt in this direction here.

It is proved in (Bachman and Narici [66], p. 396) that if T be a completely continuous linear operator on a normed linear space X, then for $\lambda \neq 0$, the null space $N(\lambda - T)$ of $\lambda - T$ is finite dimensional. We prove the following corresponding results for strictly singular operator and S-operator.

6.2 Study of the nature of null space of $\lambda - T$, T being an S-operator on X.

We study the nature of the null space of $\lambda - T$ where T is an S-operator. We prove the following theorem :

Theorem 6.1 : Let T be an S-operator on a non-normable F-space X . then for $\lambda \neq 0$, the null space, $N(\lambda - T)$ of $\lambda - T$, is Montel space.

Proof: Since

$$T : X \to X$$
$$\lambda - T : X \to X$$

Let $N(\lambda - T) = U$. Since λ is an eigen value of $T \implies$

$$(\lambda - T)x = 0 \quad , \quad \text{for all } x \in U$$

$$\implies (\lambda - T)U = 0$$
$$\implies TU = \lambda U$$

which implies that

$$T : X \to X$$

such that

$$U \to \lambda U$$

$\Longrightarrow$ the restriction of T to U is a homeomorphism. It is easy to see that U is a closed subspace of X. Since T is an S-operator, U is a Montel space. This completes the proof of the theorem.

Remark 6.1 : It immediately follows from Theorem 6.1 that if T is strictly singular operator, then $N(\lambda - T)$ is of finite dimension.

REFERENCES

Bachman [66].

APPENDIX (A)

A NOTE ON HURWITZ'S THEORM

A.1 Introduction :

The appendix as a paper entitled " A Note on Hurwitz's Theorem " has been published in " Mathematics Student " Vol. XXXIV No.2 April - June 1966.

In appendix we study the connection between the zeros of a sequence of analytic functions and limit function. Hurwitz proved Theorem 0.31 in Math. Annalen 33(1889) pp. 246-266 It is also stated in [Copson, [35] p. 121], establishing the connection between the zeros of a sequence of analytic functions and its limit function.

A.2 Connection between the zeros of a sequence of analytic functions and its limit function :

We establish the connection between the zeros of a sequence of analytic functions and its limit function under weaker hypothesis. We prove two theorems establishing the connection between the zeros of a sequence of analytic functions and its limit function and give several examples.

We state and prove the first theorem.

Theorem A.1 : If $\{f_n(z)\}$ is a sequence of regular function inside a simple closed contour C, and if $f_n(z)$ converges to $f(z)$ at an everywhere dense set of points in C(where $f(z)$ is not identically zero on C) then every zero of $f(z)$ is a zero of $f_n(z)$ for sufficiently large values of n or else it is a limiting of the zeros of

$f_n(z)$ inside C, with a possible exception of a non-dense set of points in C at which $f(z)$ is not regular.

Proof : The function $|f_n(z)|$ can be expressed by $\chi(x,y,t)$, a function of three variables where $t = \frac{1}{n}$. The function $\chi(x,y,t)$ is in the first instance defined only for values of t, of the form $t = \frac{1}{n}$ but it can be extended to the case in which t has all values in the interval $o \leq t \leq 1$, by the following rule when t lies in the interval $(\frac{1}{n+1}, \frac{1}{n})$

$$\chi(x,y,t) = \chi(x,y,t) + \frac{\frac{1}{n} - t}{\frac{1}{n} - \frac{1}{n+1}} \left[\chi(x,y,\frac{1}{n+1}) - \chi(x,y,\frac{1}{n}) \right]$$

$$\chi(x,y,o) = \lim_{n \to \infty} \chi(x,y,\frac{1}{n}) .$$

Now the function $\chi(x,y,t)$ so defined in three dimensional space is continuous with respect to t for every fixed value of x and y. By hypothesis $\left\{f_n(z)\right\}$ is a sequence of regular functions. Therefore it is continuous with respect to (x,y) for every fixed value of n, consequently the function $\chi(x,y,t)$ will be continuous with respect to x and y for every fixed value of t. Since the function $\chi(x,y,t)$ of three variables x,y,t is everywhere continuous with respect to each variable, it follows by Theorem 0.32 that $\chi(x,y,t)$ is atmost a pointwise discontinuous function relatively to three dimensional domain hence is bounded at everywhere dense set of points with respect (x,y,t) and consequently $|f_n(z)|$ will be uniformly bounded at everywhere dense set of points in C, we can always find a subdomain D_1 of domain D bounded by C, such that $|f_n(z)|$ will be uniformly bounded in D_1. By Theorem 0.33 $f_n(z)$ converges uniformly to f(z) in D_1, hence analytic function in D_1. Now the theorem follows from Theorem 0.31 in D_1. Consequently the theorem follows in C with the exception of non-dense set in C. This completes the proof of the theorem.

We state and prove the second theorem

Theorem A.2 : If $\{f_n(z)\}$ is a sequence of analytic functions, regular inside a simple closed contour C, with the exception of a non-dense set in C and $f_n(z)$ converges to f(z) at everywhere dense set in C (where f(z) is not identically zero on C), then every zero of f(z) is a zero of $f_n(z)$ for sufficiently large values of n or else it is limiting point of the zeros of $f_n(z)$ inside C, with the exception of the non-dense set in C, at which f(z) is not regular.

Proof : The proof follows on similar argument as that of Theorem A-1.

We give seven examples here :

Examples A.1 (a) (i) $f_n(z) = n \sin \left(\frac{\tan z}{n}\right)$

(ii) $f_n(z) = nz \sin \left(\frac{e^{\tan z}}{n}\right)$

(b)(iii) $f_n(z) = n \sin \left(\frac{\tanh z}{n}\right)$

(iv) $f_n(z) = nz \sin \left(\frac{e^{\tanh z}}{n}\right)$

where D is the closed and bounded domain bounded by a simple closed contour C which contains (a) the zeros and poles of $\tan z$ (b) the zeros and poles of $\tan hz$

(v) $f_n(z) = (z + \frac{1}{n}) \frac{1}{P_m(z)}$, where $P_m(z)$ is a polynomial of degree m (fixed positive integer), D is the closed and bounded domain bounded by a simple closed contour C which contains the zeros of $P_m(z)$ and the zero of the limiting function is a not a zero of $P_m(z)$.

(vi) $f_n(z) = n \sqrt{Q_{m_1}(z)} \sin\left(\frac{\sin z}{n}\right)$

where $Q_{m_1}(z)$ is a polynomial of degree m_1 (fixed positive integer), D is the closed and bounded domain bounded by a simple closed contour C which contains the zeros of $\sin z$ and $Q_{m_1}(z)$ and the zeros of $Q_{m_1}(z)$ are not the zeros of $\sin z$.

Remark A.1 : In the above examples theorem A.1 and Theorem A.2 hold. Example (vi) shows that $f(z)$ need not single-valued in Theorem A.2.

(vii) $f_n(z) = \frac{e^{\cot z}}{n} \cos z$

where D is the closed and bounded domain bounded by a simple closed contour C which contains the zeros of $\sin z$ and $\cos z$.

Remark A.2 : In example (vii) Theorem A.1 and Theorem A.2 do not hold, because, the condition of the theorems are not satisfied.

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Copson [35], Hobson [27], Titchmarsh [32].

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